SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 2

SOLUTIONS

Problem 1. Consider the transformation group $\Gamma = \left\{R_{k\pi/2} : k \in \mathbb{Z}\right\}$, where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation determined by the matrix $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that the quotient space $\mathbb{R}^2/\Gamma$ is a topological manifold. Is it homeomorphic to a familiar one? Furthermore, does it have a smooth structure such that the map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is a submersion? Does there exist a smooth structure such that π is C^∞? Justify your answers through pictures and "moral" arguments: do not write formal proofs.

Solution. Come talk to me in office hours and we can draw some fun pictures! □

Problem 2. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, and assume that $D \subset \mathbb{T}^2$ is a discrete subgroup. Show that $\mathbb{T}^2/D$ is diffeomorphic to $\mathbb{T}^2$. [*Hint*: Show that the lift of D to $\mathbb{R}^2$ is also a discrete subgroup containing $\mathbb{Z}^2$. You may use the fact that all discrete subgroups of $\mathbb{R}^2$ are isomorphic to $\{0\}$ $\mathbb{Z}$, and $\mathbb{Z}^2$.]

Solution. Let $D \subset \mathbb{T}^2$ is a discrete subgroup and $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the factor map (ie, the map which assigns to a vector v its coset $[v] = v + \mathbb{Z}^2 \subset \mathbb{R}^2$. Since D is both discrete and compact (being a subset of a compact space), it is finite. Then $\tilde{D} = \pi^{-1}(D)$ is also a discrete subgroup, since if $\tilde{D}$ accumulated at a point in $\mathbb{R}^2$, $D = \pi(\tilde{D})$ would have an accumulation point as well. Furthermore, it is a compact extension of $\mathbb{Z}^2 = \pi^{-1}(0)$ since D is finite. It is also torsion-free, since it is a subgroup of $\mathbb{R}^2$. Therefore, by the classification of finite rank abelian groups, it must be isomorphic to $\mathbb{Z}^2$ itself. Choose generators v_1 and v_2 for $\tilde{D}$, and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $L(t, s) = tv_1 + sv_2$. Then $L(\mathbb{Z}^2) = \tilde{D}$ by construction. We claim that L induces a diffeomorphism between $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{R}^2/\tilde{D} = \mathbb{T}^2/D$.

Indeed, the map $\bar{L}(v + \mathbb{Z}^2) = L(v) + \tilde{D}$ is well-defined, since a different representative of v in $\mathbb{T}^2$ must differ by an element of $\mathbb{Z}^2$, which is taken to an element of $\tilde{D}$ by L. Furthermore, since the local charts for $\mathbb{T}^2$ are constructed by lifting a neighborhood to $\mathbb{R}^2$, and similarly for $\mathbb{T}^2/\tilde{D}$, in the canonical local charts on $\mathbb{R}^2$ (which are just given by the identity), the map is linear, and hence C^∞. □

Problem 3. Let $\varphi : U \to \mathbb{R}^n$ be a smooth chart for a smooth manifold M. Define a corresponding set $\hat{U} = \bigcup_{p \in U} T_pM \subset TM$, and let an element of $\hat{U}$ be denoted by v_p, where p denotes the basepoint of the vector v. Define $\hat{\varphi} : \hat{U} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\hat{\varphi}(v_p) = (\varphi(p), v_{p,\varphi}),$$

where $v_{p,\varphi} \in T_p^\varphi M := T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n$ is the vector v as represented in the chart φ (recall that T_pM is isomorphic to each $T_p^\varphi M$ for any chart φ whose domain contains p). Show that if $\mathcal{A}$ is a smooth atlas of charts for M, then $\hat{\mathcal{A}} = \left\{(\hat{U}, \hat{\varphi}) : (U, \varphi) \in \mathcal{A}\right\}$ is a smooth atlas on TM (this is the *smooth atlas on TM induced by $\mathcal{A}$*).

Solution. We use the notations as described in the problem. To show that the collection $\hat{\mathcal{A}}$ forms a smooth atlas for TM, the transition maps are C^∞. First, notice that $\hat{U}$ and $\hat{V}$ intersect if and only if U and V intersect on M, since if a vector lies in the intersection, its base point must lie in both U and V. Let $(\hat{U}, \hat{\varphi})$ and $(\hat{V}, \hat{\psi})$ be charts for TM and $W = U \cap V$. Then consider the map $\hat{\varphi} \circ \hat{\psi}^{-1} : \hat{\psi}(W) \to \mathbb{R}^n \times \mathbb{R}^n$. We compute the derivative of $\hat{\varphi}$ directly. Notice that moving the vector component does not change the basepoint, so the derivative by moving along the n coordinates does not change the first n. However, the derivative by moving along first n coordinates can change both the position on the output, as well as the vectors. Indeed, the total derivative is the block matrix:

$$D(\hat{\varphi} \circ \hat{\psi}^{-1}) = \begin{pmatrix} D(\varphi \circ \psi^{-1}) & \mathbf{0} \\ A & D(\varphi \circ \psi^{-1}) \end{pmatrix}$$

where each block is an $n \times n$ matrix, and A is the matrix of second derivatives $A_{ij} = \dfrac{\partial D(\varphi \circ \psi^{-1})_{ij}}{\partial x_i}$, since it measures how much the (i,j)th component derivative changes as the base component moves. This map is locally invertible, and hence a diffeomorphism, since its derivative is a block lower triangular matrix with invertible diagonal blocks, and hence invertible. □

Remark 1. We have only discussed C^∞ structures, but you can talk about manifolds with C^r-structures, with $r \geq 1$. You can imagine what the definition is. In this setting, C^r-manifolds have a canonical C^{r-1}-structure on TM. Of course, when $r = \infty$, $r - 1 = \infty$ as well.

Problem 4. Show that if $U \subset \mathbb{R}^k$ is open, $\varphi : U \to \mathbb{R}^n$ is C^∞ and $d\varphi(x)$ is injective, then there exists a C^∞ change of coordinates diffeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ defined near $\varphi(x)$ such that $H \circ \varphi$ takes values in $\mathbb{R}^k \subset \mathbb{R}^n$.

Solution. Let $E \subset \mathbb{R}^n$ denote the image of $d\varphi(x) : \mathbb{R}^k \to \mathbb{R}^n$. By injectivity, $\dim(E) = k$. Let $L : \mathbb{R}^{n-k} \to \mathbb{R}^n$ be any linear map such that $\mathbb{R}^n$ is the direct sum of E and the image of L. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, y) = \varphi(x) + Ly$, where $(x, y) \in \mathbb{R}^n$ is written componentwise with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$. Notice that

$$DF(x, y) = (d\varphi(x) \ L)$$

So that the image of $DF(x, 0)$ is the sum of the images of $d\varphi(x)$ and L, which is all of $\mathbb{R}^n$. Hence $DF(x, y)$ is an isomorphism, and there exists open neighborhoods in $\mathbb{R}^n$ for which F has an inverse. Let $H : \mathbb{R}^n \to \mathbb{R}^n$ denote the inverse of F. Then $(x, 0) = H(F(x, 0)) = H(\varphi(x) + L \cdot 0) = H \circ \varphi(x)$. □

Problem 5. Let M be a C^∞ manifold, $k \in \mathbb{N}$ and $X \subset M$ be a closed, connected subset such that for every $x \in X$, there exists $U \subset \mathbb{R}^k$ and an embedding $\varphi : U \to M$ such that the image of φ is an open neighborhood of x in X. Show that X has a C^∞ manifold structure such that the inclusion of X into M is an embedding. [*Hint*: Use the previous problem!]

Proof. Since M is a closed connected subset of a manifold, it is second countable and Hausdorff. By assumption is locally Euclidean (the charts are the inverses of the maps φ. So we must verify the transition maps are C^∞. Let φ and ψ denote the inverses of maps provided in the assumption of the problem, $U_\varphi, U_\psi \subset X$ denote their domains (open subsets of $X \subset M$) and $V_\varphi, V_\psi \mathbb{R}^k$ denote their ranges. Let $x \in U_\varphi \cap U_\psi$, and choose a C^∞ chart of x for M, $\eta : U_\eta \to V_\eta$, so that $U_\eta \subset M$ is open and $V_\eta \subset \mathbb{R}^n$ is open.

We begin by restricting maps appropriately. Indeed, notice that $U_\varphi \cap U_\psi$ is an open in X. It is therefore the intersection of an open neighborhood of M with X. By intersecting it with U_η, we may assume that it is in fact equal to U_η. In summary, to study differentiability properties at x, we may assume that $U_\varphi = U_\psi = U_\eta \cap X$.

Observe that $\eta \circ \varphi^{-1}$ is a C^∞ map defined locally from $\mathbb{R}^k$ to $\mathbb{R}^n$ and since φ^{-1} is an embedding, $d(\eta \circ \varphi^{-1})$ is injective at $\varphi(x)$. Hence, by the previous problem, we may replace η by $H \circ \eta$ and assume that image of $\eta \circ \varphi^{-1}$ is contained in $\mathbb{R}^k \times \{0\}$. However, since we assume that ψ and φ have the same domain, ψ^{-1} and φ^{-1} have the same image. Hence $\eta \circ \psi^{-1}$ must also be contained in $\mathbb{R}^k \times \{0\}$.

Finally, note that since $\eta \circ \varphi^{-1}$ and $\eta \circ \psi^{-1}$ are full rank, C^∞ maps from open subsets of $\mathbb{R}^k$ to open subsets of $\mathbb{R}^k \times \{0\} \cong \mathbb{R}^k$, they have local inverses by the inverse function theorem. Therefore, $\varphi \circ \psi^{-1} = (\eta \circ \varphi^{-1})^{-1} \circ (\eta \circ \psi^{-1})$ is a local C^∞ diffeomorphism at $\psi(x)$. □